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A Survey of Fixed Point and Economic Game Theory

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Abstract

This paper surveys the development of fixed point theory regarding to game theory. Moreover, we focus on the theoretical results applied for economics. Many recent papers are also collected and summarized throughout a particular period of time.

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1 INTRODUCTION

According to researches on fixed point theory, its development has been rapidly growing and playing an important role in modern mathematics. As in most situation, the fixed point problem is usually considered in various ways. This theory shows how the pure relates with applied mathematics. Therefore, it is used in solving other branches of mathematics, for instance, variation and optimization problems, partial differential equations and probability problems. Also, many mathematicians go forwards for searching the applications of these results in such diverse fields as biology (see [12]), chemistry (see [8]), economics (see [3]), game theory (see [13]), etc.

Game theory is the study behaviors of players - people who is in strategic situations - what to do under decision's other players have effect. So, similar to a chess game, there is a set of players, a set of strategies available to those players and a range of payoffs of each integration of strategies. Furthermore, it becomes now a standard tool in economics. Economists then use game theory to explain, predict how people behave. They have used it to study auctions, bargaining, merger pricing, oligopolies and much else. Contributions to game theory are constructed by economists through the different fields and interests, and economists commonly collect results in game theory with work in other areas. One of those, the theory of equilibrium, has presently an extensive practicability in such game theory. Its importance has been proved by awarding the Nobel Prize for Economics to K. Arrow in 1972, G. Debreu in 1983, J. Nash, J. Harsanyi and R. Selten in 1994, and R.J. Aumann and T.C. Schelling in 2005 for applying the theory of games in economy.

The propose of this paper briefly shows the collected fixed point theorems applied in game theory. We begin with an overall image of the evolution of fixed point theory, after that, we emphasize on its integration with economics. Finally, there exists a summary of research directions in this area.

2 FIXED POINT PROBLEMS

Fixed point theorems require maps f of a set X into itself under certain conditions which guarantee an existence theorem and a uniqueness theorem - how there exists a fixed point for a mapping and also it is a unique point.

Definition 2.1. Let $f : X \to X$ be a mapping, and if there exists $\overline{x} \in X$ such that $f(\overline{x}) = \overline{x}$, then $\overline{x}$ is called a **fixed point** (fix-point) of f.

2.1 Topological Fixed Point Theory

In 1912, L.E.J. Brouwer proved a fixed point theorem which is in the history of topology with applications such that it is principally a elementary theorem to game theory, for example, in Nash equilibrium.

Theorem 2.2. [4] Let X be a nonempty, convex, compact subset of $\mathbb{R}^n$, and let $f : X \to X$ be a continuous function from X to itself. Then f has a fixed point.

2.2 Metric Fixed Point Theory

Metric fixed point theory becomes a famous tool of a scientific area because the fundamental result of Banach in 1922.

Several research areas of mathematics and other sciences are attempted to relate with applications of such results.

Theorem 2.3. [2] Let (X, d) be a complete metric space and $f : X \to X$ be a contractive mapping, that is, there exists $k \in [0, 1)$ such that $d(f(x), f(y)) \leq kd(x, y)$ for all $x, y \in X$. Then we have the following:

1. The mapping f has a unique fixed point $\overline{x} \in X$;
2. For each $x_0 \in X$, the sequence $\{x_n\}$ defined by $x_{n+1} = f(x_n)$ for each $n > 0$ converges to the fixed point $\overline{x}$ of f, that is, $f(\overline{x}) = \overline{x}$.

2.3 Discrete Fixed Point Theory

Tarski's fixed point theorem was stated in 1955. His result was in its most general form. Moreover, it is extended to have many important results.

Theorem 2.4. [11] If f is a monotone function (an order-preserving function or isotone), that is, $a \leq b$ implies $f(a) \leq f(b)$, on a nonempty complete lattice, then the set of fixed points of f forms a nonempty complete lattice.

3 GAME THEORY AND ECONOMICS

Game theory is the study of logical analysis of conflict and cooperation situations. Therefore, it is the explanation of how players would react in games rationally. Every player also need the maximum payoff as possible at the end of the game. However, the outcome is controlled by some condition. In the same way, the outcome output of player's actions does not depend on only their own choice alone but also get results from the other players' actions. Then, this is the reason that conflict and cooperation can be happened. A **game** is defined to be any situation in which

1. The number of **players** who may be an individual, but it may also be a more general entity like a company, a nation, or even a biological species is at least two.
2. All players have their own set of **strategies** which affect how the players select the actions.
3. The **outcome** of the game is assigned by the strategies which each player chosen.
4. Each possible outcome of the game can be represented by a numerical **payoffs** of different players.

In game theory, its structure can be divided to be 2 main parts. The classical games which include mixed equilibrium, rationalizability, and knowledge. Then, the extension games consisting of bargaining, repeated games, complexity, implementation, and sequential equilibrium. We can now mathematically define a game.

Definition 3.1. A **strategic game** is $(N, X_i, \succsim_i)$ consisting of

1. A finite set of players N.

2. For each player $i \in N$, a nonempty set of actions X_i.

3. For each player $i \in N$, a preference relation $\succsim_i$ on $X = \prod_{j \in N} X_j$.

Note that a strategic game is called **finite** if X_i is finite for all $i \in N$.

3.1 A General Model

In 1950, J. Nash described the concept of the n-person game as follows.

Definition 3.2. [9] The **normal form** of an n-person game is $(X_i, \succsim_i)_{i=1}^n$, where for each $i \in \{1, 2, 3, ..., n\}$, X_i is a nonempty set of individual strategies of player i and $\succsim_i$ is the preference relation on $X := \prod_{i \in I} X_i$ of player i.

Note that the individual preferences $\succsim_i$ are often represented by **utility functions**, that is, for each $i \in \{1, 2, 3, ..., n\}$ there exists a real valued function $u_i : X := \prod_{i \in I} X_i \to \mathbb{R}$, such that

$$x \succsim_i y \text{ if and only if } u_i(x) \geq u_i(y)$$

for all $x, y \in X$. Therefore, the normal form of n-person game can be written as $(X_i, u_i)_{i=1}^n$ also.

Moreover, an equilibrium of such game is defined and it is well-known as Nash equilibrium.

Definition 3.3. [9] The **Nash equilibrium** for the normal game is a point $\overline{x} \in X$ which satisfies for each $i \in \{1, 2, 3, ..., n\}$,

$$u_i(\overline{x}) \geq u_i(\overline{x}_{-i}, x_i)$$

for each $x_i \in X_i$ where $\overline{x}_{-i} = (\overline{x}_1, \overline{x}_2, ..., \overline{x}_{i-1}, \overline{x}_{i+1}, ..., \overline{x}_n)$.

3.2 An Economic Model

The situation that there are n agents who produce and sell m goods. Assume that m is the number of production units. Let $A_i \subseteq \mathbb{R}^l$ be a set of plans each agent use. In each production unit $j \in \{1, 2, 3, ..., m\}$, the activity is organized according to a production plan $d_j \in \mathbb{R}^l$. Agents are both producers and consumers. We have

$$\alpha_{ji} \geq 0, \forall i \in \{1, 2, ..., n\}, \Sigma_{i=1}^n \alpha_{ji} = 1$$

for each j $\in \{1, 2, ..., l\}$

The preference relation of the consumer i on the consumption plans set A_i is denoted by $\succsim_i$ and assume that is represented by the utility function u_i.

Definition 3.4. [1] An **economy** ε is represented as

$$\varepsilon = \{(A_i)_{i=1}^n, (D_j)_{j=1}^m, (w_i)_{i=1}^n, (\succsim_i)_{i=1}^n, (\alpha_{ji})_{j,i=1}^{m,n}\}.$$

where P represents the set of all normalized price systems.

Denote by A and D the sets of complete consumption plans, respectively, production plans, i. e., $A := \prod_{i=1}^n A_i$, $D := \prod_{j=1}^m D_j$ and by $A^+ = \Sigma_{i=1}^n A_i$, $D^+ = \Sigma_{j=1}^m D_j$.

For a given price system $p \in P$, and a complete production plan $d = (d_1, d_2, ..., d_m) \in D$, the budget set of agent i is defined as

$$B_i(p, d) = \{\alpha_i \in A_i : \langle p, a_i \rangle \leq \langle p, \omega_i \rangle + \Sigma_{j=1}^m \alpha_{ji} \langle p, d_j \rangle\}.$$

Definition 3.5. [1] A **competitive equilibrium** of ε is $(a^*, d^*, p^*) \in A \times D \times P$ satisfy the following conditions

1. For each $j \in \{1, 2, ..., m\}, \langle p^*, d_j^* \rangle \geq \langle p^*, d_j \rangle$ for all $d_j \in D_j$.
2. For each $i \in \{1, 2, ..., n\}, a_i^* \in B_i(p^*, d^*)$ and $a_i^* \succsim_i a_i$ for all $a_i \in B_i(p^*, d^*)$
3. $\Sigma_{i=1}^n a_i^* \leq \Sigma_{i=1}^n \omega_i + \Sigma_{j=1}^m d_j^*$
4. $\langle p^*, \Sigma_{i=1}^n a_i^* - \Sigma_{i=1}^n \omega_i - \Sigma_{j=1}^m d_j^* \rangle = 0.$

Condition 4. says that prices become 0 when the offer is higher than the demand.

After that, some constraint correspondences have been considered.

Definition 3.6. [6] An **abstract economy** $\Gamma = (X_i, A_i, u_i)_{i=1}^n$ is defined as a family of n ordered, where $A_i : X := \prod_{i=1}^n X_i \to 2^{X_i}$ are correspondences and $u_i : X \times X_i \to \mathbb{R}$.

G. Debreu stated the definition of equilibrium in 1952 which it is a natural extension of equilibrium introduced by J. Nash.

Definition 3.7. [6] An **equilibrium** for Γ is a point $\overline{x} \in X$ which satisfies for each $i \in \{1, 2, 3, ..., n\}$,

$$\overline{x}_i \in A_i(\overline{x}) \text{ and } u_i(\overline{x}) \geq u_i(\overline{x}_{-i}, x_i)$$

for each $x_i \in A_i(\overline{x})$.

In 1975, W. Shafer and H. Sonnenschein proposed a model of abstract economy with a finite set of agents. Each agent has a constraint correspondence A_i and, instead of the utility function u_i, they have a preference correspondence P_i. This model generalizes G. Debreu's model, whereas, using the utility functions, one can define the preference correspondences as follows

$$P_i(x) := \{y_i \in A_i(x) : u_i(x, y_i) > u_i(x, x_i)\}.$$

Then the condition of maximizing the utility function to obtain the equilibrium point becomes

$$A_i(x) \cap P_i(x) = \emptyset \text{ for each } i \in \{1, 2, ..., n\}.$$

W. Shafer and H. Sonnenschein's model can be described as follows:

Definition 3.8. [10] Let the set of agents be the finite set $1, 2, ..., n$. For each $i \in \{1, 2, ..., n\}$, let X_i be a nonempty set. An **abstract economy** $\Gamma = (X_i, A_i, P_i)_{i=1}^n$ is defined as a family of n ordered, where for each $i \in I$

1. $A_i : X := \prod_{i=1}^n X_i \to 2^{X_i}$ is a constraint correspondence
2. $P_i : X := \prod_{i=1}^n X_i \to 2^{X_i}$ is a preference correspondence.

An equilibrium for W. Shafer and H. Sonnenschein's model is defined as follows

Definition 3.9. [10] An **equilibrium** for Γ is a point $\overline{x} \in X := \prod_{i=1}^n X_i$ which satisfies for each $i \in \{1, 2, 3, ..., n\}$,

$$\overline{x}_i \in A_i(\overline{x}) \text{ and } A_i(\overline{x}) \cap P_i(\overline{x}) = \emptyset$$

for each $x_i \in A_i(\overline{x})$.

4 FIXED POINT THEORY VIA GAMES

Since Kakutani's fixed point theorem extends Brouwer's Theorem to set-valued functions. Then, we recall a definition of a fixed point for a multivalued mapping (or correspondence).

Definition 4.1. Let $\mathcal{F}(X)$ be the family of all closed convex subsets of X. A point mapping $x \mapsto \varphi(x) \in \mathcal{F}(X)$ of X into $\mathcal{F}(X)$ is called **upper semicontinuous** if

$$x_n \to x_0, y_n \in \varphi(x_n) \text{ and } y_n \to y_0$$
$$\text{imply } y_0 \in \varphi(x_0)$$

It is easy to see that this condition is equivalent to saying that the graph of $\varphi(x) : \Sigma_{x \in X} x \times \varphi(x)$ is a closed subset of $X \times X$.

Definition 4.2. A point $\overline{x} \in X$ is said to be a **fixed point** of the multivalued mapping F if $\overline{x} \in F(\overline{x})$.

Therefore the general fixed point theorem can be stated as

Theorem 4.3. [14] Let X be a nonempty, convex, compact subset of $\mathbb{R}^n$, and let $F : X \to 2^X$ be an upper semicontinuous, nonempty-valued, closed-valued, and convex-valued correspondence. Then F has a fixed point.

This leads to illustrate how fixed point theorems adapted in game theory.

Theorem 4.4. [14] The strategic game $(N, X_i, \succsim_i)$ has a Nash equilibrium if X_i is a nonempty, compact, convex subset of a Euclidean space and $\succsim_i$ is continuous and quasi-concave on X_i for all $i \in N$.

Fixed point theorems on such mappings constitute one of the most important arguments in the fixed point theory of correspondences.

Definition 4.5. Let X and Y be any sets. The **graph** of a correspondence $F : X \rightrightarrows Y$, denoted $Gr(F)$, is the set

$$Gr(F) := \{(x, y) \in X \times Y : y \in F(x)\}.$$

Another important kind of correspondence in fixed point theory is the class of closed correspondences. So, there is an important property for correspondences.

Definition 4.6. A correspondence $F : X \rightrightarrows Y$ is **closed** if it has a closed graph, i.e., $Gr(f)$ is a closed subset of $X \times Y$.

Many correspondences have been improved in reaching some new results in game theory along with fixed point theorems, namely, L_S-majorized, $\mathcal{U}$-majorized, F_θ-majorized, etc.

Definition 4.7. [7] Let X be a topological space, and Y be a nonempty subset of a vector space E, $\theta : X \to E$ be a mapping and $\phi : X \rightrightarrows Y$ be a correspondence, then

1. ϕ is said to be of **class** Q_θ (or Q) if
   a. for each $x \in X$, $\theta(x) \notin \mathrm{cl}\phi(x)$
   b. ϕ is lower semicontinous with open and convex values in Y
2. ϕ_x is a Q_θ**-majorant** of ϕ at x, if there is an open neighborhood $N(x)$ of x in X and $\phi_x : N(x) \rightrightarrows Y$ such that
   a. for each $z \in N(x)$, and $\phi(z) \subset \phi_x(z)$ and $\theta(z) \notin \mathrm{cl}\phi_x(z)$
   b. ϕ is lower semicontinous with open and convex values;

3. ϕ is said to be **Q_θ-majorized** if for each $x \in X$ with $\phi(x) \neq \emptyset$, there exists a Q_θ-majorant ϕ_x of ϕ at x.

Liu and Cai did not only define Q_θ-majorized but they also gave the result of an existence of a maximal element in 2001.

Theorem 4.8. [7] Let X be a paracompact convex subset of a Hausdorff locally convex topological vector space E, D a nonempty compact metrizable subset of X. Let $P : X \rightrightarrows D$ be Q_θ-majorized, then there exists a point $x \in X$ such that $P(x) = \emptyset$.

Moreover, an existence of equilibria in abstract economy are proved as well.

Theorem 4.9. [7] Let $\Gamma = (X_i, A_i, B_i, P_i)_{i\in I}$ be an abstract economy where I is any (countable or uncountable) set of agents such that for each $i \in I$

1. X_i is a nonempty convex subset of Hausdorff locally topological vector space E_i, $X := \prod_{i\in I} X_i$ is paracompact, D_i is nonempty compact metrizable subset of X_i
2. A_i, B_i, P_i are correspondences $X \rightrightarrows D_i$, for each $x \in X$, $A_i(x)$ is nonempty, B_i is lower semicontinuous and convex closed valued, and $\text{cl}B_i(x) \subset D_i$
3. The set $E^i = \{x \in X, A_i(x) \cap P_i(x) \neq \emptyset\}$ is closed in X
4. The mapping $A_i \cap P_i : X \rightrightarrows D_i$ is Q_θ-majorized,

Then Γ has an equilibrium point, i.e., there exists a point $x^* \in X$ such that for each $i \in I$, $x_i^* \text{cl}B_i(x^*)$ and $A_i(x^*) \cap P_i(x^*) = \emptyset$.

Next, it is L_S-majorized correspondence and its results which is introduced in book of KKM theory and applications in nonlinear analysis.

Definition 4.10. [15] Let $A_i : X \rightrightarrows Y_i$ be a correspondence for each $i \in I$. Then A_i is said to be

1. of **class $\boldsymbol{L_S}$** if
   a. A_i is convex valued
   b. $y_i \notin A_i(S(y))$ for each $y \in Y$
   c. $A_i^{-1}(y_i) := \{x \in X : y_i \in A_i(x)\}$ is open in X for each $y_i \in Y_i$
2. **$\boldsymbol{L_S}$-majorized** if for each $x \in X$, there exists an open neighborhood $N(x)$ of x in X and a convex-valued mapping $B_x : X \rightrightarrows Y_i$, which is called an L_S-majorant of A_i at x, such that
   a. $A_i(z) \subset B_x(z)$ for each $z \in N(x)$
   b. $y_i \notin B_x(S(y))$ for each $y \in Y$
   c. $B_x^{-1}(y_i)$ is open in X for each $y_i \in Y_i$.

Theorem 4.11. [15] Let X be a compact Hausdorff topological space and Y be a nonempty convex subset of a Hausdorff topological vector space E, $S : Y \to X$ be continuous and $A : X \rightrightarrows Y$ be L_S-majorized. Then there exists $x \in X$ such that $A(x) = \emptyset$.

There is a theorem in game theory using L_S-majorized correspondence in 2006 stated by S.Y. Chang.

Theorem 4.12. [5] Let $\Gamma = \{X_i, A_i, B_i, P_i\}_{i\in I}$ be an abstract economy, where I can be an infinite set of agents, such that for each $i \in I$, the following conditions are satisfied

1. X_i is a nonempty convex subset of a Hausdorff topological vector space E_i and D is a compact subset of $X := \prod_{i\in I} X_i$
2. for each $x \in X$, $A_i(x)$ is nonempty and $\mathrm{co}A_i(x) \subset B_i(x)$
3. $F_i = \{x \in X : x_i \in \mathrm{cl}B_i(x)\}$ is closed in X
4. $A_i : X \rightrightarrows X_i$ has compactly open lower sections
5. the correspondence $A_i \cap P_i : X \rightrightarrows X_i$ is L_S-majorized in F_i
6. for each finite set $S \subset X$, there exists a compact convex set $K \prod_{i\in I} K_i$ containing S such that for each $x \in [K \backslash D]$, there exists $i \in I$ such that $(A_i \cap P_i)(x) \cap K_i \neq \emptyset$.

Then, there exists $x^* \in X$ such that $x_i^* \in \mathrm{cl}B_i(x^*)$ and $A_i(x^*) \cap P_i(x^*) = \emptyset$ for each $i \in I$.

5 RESEARCH DIRECTIONS

Now, widespread results of fixed point theorems applied in game theory are to use correspondences in sense of majorized constructing the existence of an equilibrium for a generalized in game theory. Secondly, they consider economic game and model through optimization problems. Otherwise, it is applied mathematics in computational science to reach in equilibrium problems in games.

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